MA 241 : Ordinary Differential Equations (JAN-APR, 2018)

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Problem set 2

1. Let $f(t, y, \dot{y}) = h(t)$ be the general form of the first order equation, where $h = h(t)$ is all the combined non-homogeneous terms. Consider $L(r, s) = f(t, r, s)$, where t is fixed. If L is linear in r and s, show that there exists functions $p_0 = p_0(t)$ and $p_1 = p_1(t)$ so that f takes of the form,
$$f(t, y, \dot{y}) = p_0(t)\dot{y}(t) + p_1(t)y(t).$$
More generally an n^{th} order linear equation has the general form,
$$p_0(t)y^n(t) + p_1(t)y^{n-1}(t) + ... + p_n((t)y(t) = h(t).$$

2. Give examples of linear and non-linear equations.

3. Classify the following equations as linear or non-linear:
i. $\dot{y} = ay - by^2$ ii. $\dot{y} = -t/y$ iii. $\dot{y} = -y/t$ iv. $\dot{y}(t) = sin(t)$
v. $\dot{y} = |y|$ vi. $y\dot{y} = y$ vii. $\dot{y} = \sin\ y$ viii. $y\dot{y} = \frac{g}{W}(W - B - cy)$
ix. (Duffing Equation): $\ddot{y} + \delta\dot{y} + \alpha y + \beta x^3 = 0$
x. (Van der Pol Equation): $\ddot{y} - \mu(y^2 - 1)\dot{y} + y = 0$
xi. (prey-predator system): $\dot{x} = ax - bxy,\ \dot{y} = -cy + dxy$
xii. (Epidemiology): $\dot{S} = -\beta SI,\ \dot{I} = \beta SI - \gamma I$
xiii. $siny + xcos\dot{y} = 0$ xiv. (Bernoulli Equation): $\dot{y} + \phi(t)y = \psi(t)y^n$
xv. (Reduced Bernoulli equation): $\dot{y} + (1 - n)\phi(t)y = (1 - n)\psi(t)$
xvi. (Generalized Riccati equation): $\dot{y} + \psi(t)y^2 + \phi(t) + \chi(t) = 0$

4. Consider the Bernoulli equation
$$\dot{x} + \phi x = \psi x^n$$
where ϕ, ψ are continuous functions. For $n \neq 1$, it is non-linear; show that it can be reduced to a linear equation by the substitution $y = x^{1-n}$. Then solve the equation.

5. Solve: (i). $\dot{x} + e^t x = e^t x^2$ (ii). $\dot{x} + t^n x = x^n$.

6. Consider the Generalized Riccati equation

$$\dot{y} + \psi(t)y^2 + \phi(t)y + \chi(t) = 0,$$

where ψ, χ, ϕ are functions of t. In general we do not have solutions in explicit form. Assume $x = x_1$ is one known solution, let x be any other solution. Write $x = x_1 + y$. Show that y satisfies the Bernoulli equation

$$\dot{y} + (2x_1\psi + \phi)y + \psi y^2.$$

7. Find the general solution of the equation $\dot{x} + x^2 + x - (1 + t + t^2) = 0$.

8. Assume $\psi(t) \neq 0$ for all t in the non-linear Riccati equation $\dot{y} + \psi(t)y^2 + \phi(t)y + \chi(t) = 0$ (Of course, if $\psi \equiv 0$, it is a linear equation). Make the following substitution $y = \frac{1}{\psi}\frac{\dot{z}}{z}$, reduce the non-linear Riccati equation to a second order linear equation.

9. Reduce the original Riccati (non-linear) equation, $\dot{x} + ax^2 = bt^m$; a, b constants, to a second order linear equation $\ddot{z} - abt^m z = 0$.

10. a) Consider the linear problem $\dot{y} + py = q$. Show that if $q \geq 0$, then $y \geq 0$ if it is initially so, that is if $y(0) \geq 0$. Now consider the equation $\dot{x} + px = q_1$ and $\dot{y} + py = q_2$, then compare the solutions when $q_1 \geq q_2$.

    b) Consider $\dot{x} + p_1 x = q$ and $\dot{y} + p_2 y = q$. Show that, if $p_2 \geq p_1$, $x(0) \geq y(0)$ and $y \geq 0$, then $x \geq y$.

    c) Consider the inequality $\dot{y} + py \leq q$. Derive the inequality

$$y(t) \leq \exp\left(-\int_0^t p(s)ds\right)\left[y(0) + \int_0^t q(s)\exp\left(\int_0^s p(z)dz\right)ds\right].$$

    d) Derive the Gronwall's inequality. Assume f and g are continuous real valued functions defined on the interval $[a, b]$ and $g \geq 0$. Assume $f(t) \leq c + k\int_{t_0}^t f(s)g(s)ds$, where c, k are constants, $k \geq 0$, then

$$f(t) \leq c\exp\left(k\int_{t_0}^t g(s)ds\right), \qquad t_0 \in [a, b].$$

    e) (Uniqueness) Let p, q are continuous functions on $[a, b]$. Show that the linear initial value problem $\dot{x} + p(t)x = q(t), x(t_0) = x_0$ has atmost one solution.